Chapter 7.

Calculus of trigonometric functions.

In the previous chapter, when wanting to determine the derivative of e^x, we returned to the basic definition:

$$\text{If } y = f(x) \text{ then } \frac{dy}{dx} = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

Indeed, question 5 of Miscellaneous Exercise Six also required us to return to this **first principles** approach to show that if $y = x^2 + 3x$ then $\frac{dy}{dx} = 2x + 3$.

Rather than use this first principles approach every time we tend to use it to identify patterns and establish general rules, and then apply these rules to determine derivatives as required. Let us now use this approach to determine the derivative of the trigonometric function $y = \sin x$.

Note: We will use the following identity which the Preliminary Work reminded us of:

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

Using the first principles definition, and the fact, not proved here, that

$$\lim_{x\to a} \left(f(x) \pm g(x)\right) = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x):$$

$$\begin{aligned}
\text{If } \quad y = \sin x \quad \text{then} \quad \frac{dy}{dx} &= \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h} \\
&= \lim_{h\to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h} \\
&= \lim_{h\to 0} \frac{\cos x \sin h}{h} - \lim_{h\to 0} \frac{\sin x - \sin x \cos h}{h} \\
&= \cos x \lim_{h\to 0} \frac{\sin h}{h} - \sin x \lim_{h\to 0} \frac{1 - \cos h}{h} \qquad \leftarrow\bigstar
\end{aligned}$$

Thus to determine $\frac{d}{dx}(\sin x)$ we need to investigate $\lim_{h\to 0} \frac{\sin h}{h}$

and $\lim_{h\to 0} \frac{1 - \cos h}{h}$.

Such investigations follow on the next few pages and then we will return to statement ★, above, and apply what we find out about the two limits.

$$\lim_{h \to 0} \frac{\sin h}{h}$$

To determine some limits as $h \to 0$, for example $\lim_{h \to 0} (2x + 3 + h)$, we simply substitute $h = 0$ into the expression to obtain the limit (which in the example would give $2x + 3$). However, in this case, substitution of $h = 0$ into $\frac{\sin h}{h}$ gives $\frac{0}{0}$, which is undefined, so, to determine the limit, further investigation is needed.

We could:

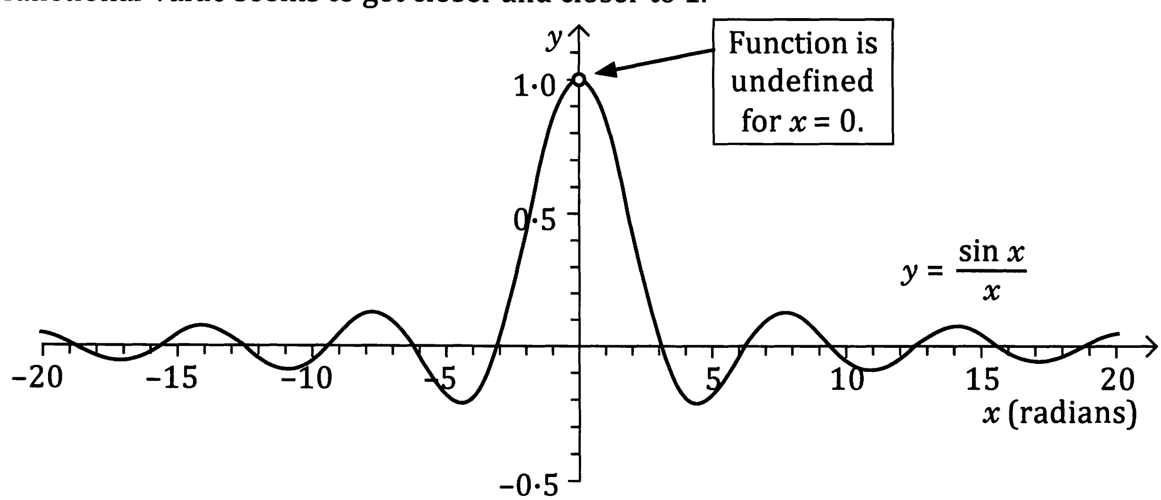
- Ask our calculator, see the display on the right (for x in radians).

$$\lim_{x \to 0} \left(\frac{\sin(x)}{x} \right) \qquad 1$$

- View the graph of $$y = \frac{\sin x}{x}$$ and see what seems to be happening as x approaches zero.
- Consider a table of values for $\frac{\sin x}{x}$ as x approaches zero.

Viewing the graph.

The graph of $y = \frac{\sin x}{x}$, for x in radians, is shown below. Notice that as we move along the graph, getting closer and closer to $x = 0$, from either the left side or the right side, the functional value seems to get closer and closer to 1.



Hence the graph of $y = \frac{\sin x}{x}$ supports the calculator statement, i.e. that $\lim_{x \to 0} \frac{\sin x}{x} = 1$, (and hence that $\lim_{h \to 0} \frac{\sin h}{h} = 1$).

Note • The graph shows an "open circle" at $x = 0$ because $\frac{\sin x}{x}$ is undefined there.

However this does not stop us investigating the behaviour of $\frac{\sin x}{x}$ as x gets closer and closer to zero.

Tables of values.

Considering x approaching zero "from the left" and x approaching zero "from the right" the following tables of values can be created (calculated values shown rounded to ten decimal places):

x approaching 0 from the left.

x	$\frac{\sin x}{x}$ (values from calculator)
−0·1	0·998 334 166 5
−0·01	0·999 983 333 4
−0·001	0·999 999 833 3
−0·000 1	0·999 999 998 3
−0·000 01	1·000 000 000 0
−0·000 001	1·000 000 000 0
−0·000 000 1	1·000 000 000 0

x approaching 0 from the right.

x	$\frac{\sin x}{x}$ (values from calculator)
0·1	0·998 334 166 5
0·01	0·999 983 333 4
0·001	0·999 999 833 3
0·000 1	0·999 999 998 3
0·000 01	1·000 000 000 0
0·000 001	1·000 000 000 0
0·000 000 1	1·000 000 000 0

Once again the statement

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

appears reasonable.

Note • The fact that as x approaches zero $\frac{\sin x}{x}$ approaches 1 means that **for small angles, measured in radians, $\sin x \approx x$.**

For example:

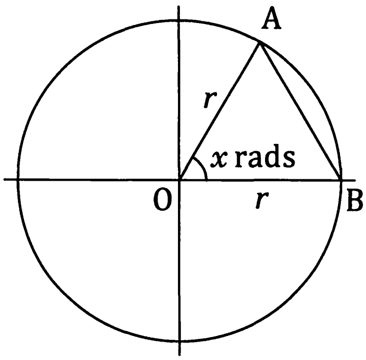
$$\sin 0{\cdot}01 = 0{\cdot}009\,999\,833\,3 \approx 0{\cdot}01.$$
$$\sin 0{\cdot}025 = 0{\cdot}024\,997\,395\,9 \approx 0{\cdot}025.$$
$$\sin 0{\cdot}0084 = 0{\cdot}008\,399\,901\,2 \approx 0{\cdot}0084.$$

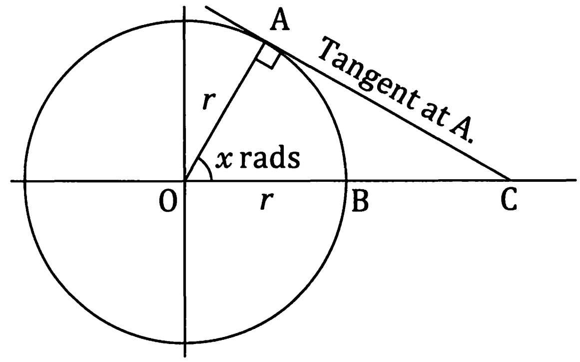
(The reader should confirm these values on a calculator.)

• A proof of $\lim_{x \to 0} \frac{\sin x}{x} = 1$ is shown on the next page.

Proof.

Consider the following circles of radius r, with $0 < x < \pi/2$.





Area ΔOAB < Area sector OAB

$$\frac{1}{2}r^2 \sin x < \frac{1}{2}r^2 x$$

$$\sin x < x$$

$$\frac{\sin x}{x} < 1 \qquad \leftarrow ①$$

Area sector OAB < Area ΔOAC

$$\frac{1}{2}r^2 x < \frac{1}{2}(\text{OA})(\text{AC})$$

$$\frac{1}{2}r^2 x < \frac{1}{2}r(r\tan x)$$

$$x < \frac{\sin x}{\cos x}$$

$$\cos x < \frac{\sin x}{x} \qquad \leftarrow ②$$

Combining ① and ②: $\cos x < \dfrac{\sin x}{x} < 1$

Writing $x \to 0^+$ for x tending towards zero from the positive side:
As $x \to 0^+$ $\quad \cos x \to \cos 0 = 1$

Thus as $x \to 0^+$, $\dfrac{\sin x}{x}$ is "sandwiched" between $\cos x$, which is approaching 1, and 1 itself.

Therefore $\dfrac{\sin x}{x}$ must also approach 1 as $x \to 0^+$.

Similar reasoning can be used to show that $\dfrac{\sin x}{x} \to 1$ as $x \to 0^-$.

Thus $\lim\limits_{x\to 0} \dfrac{\sin x}{x} = 1$, as required.

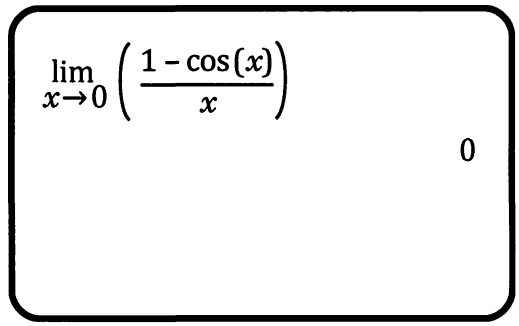
$$\boxed{\lim_{x\to 0} \frac{\sin x}{x} = 1}$$

$$\lim_{h \to 0} \frac{1 - \cos h}{h}$$

Substitution of $h = 0$ into $\frac{1 - \cos h}{h}$ gives $\frac{0}{0}$ and so, as before, to determine the limit we need to investigate further.

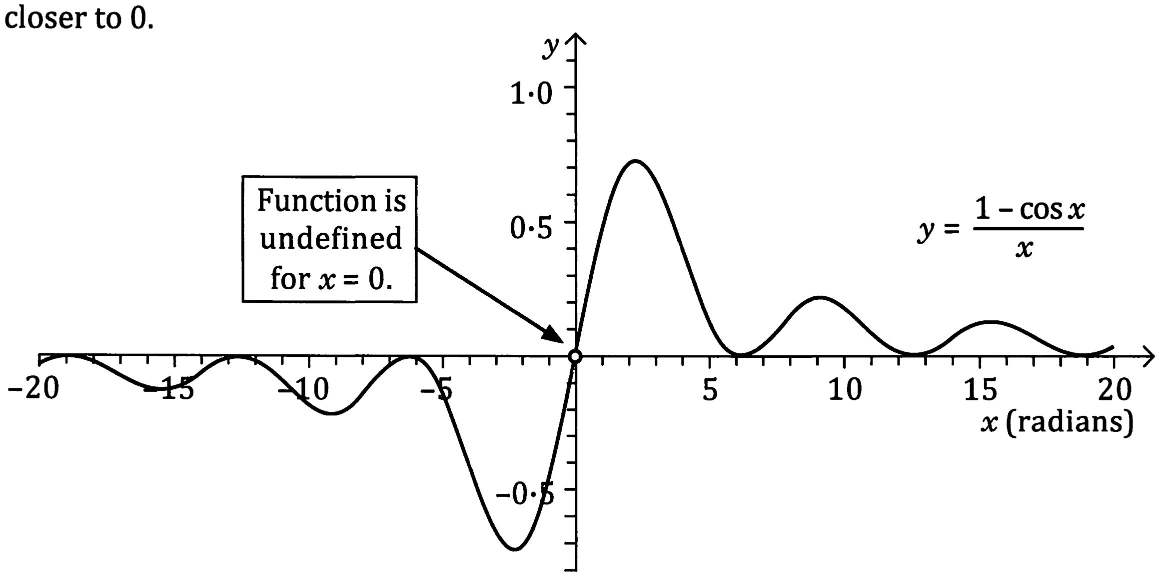
According to the display on the right

$$\lim_{x \to 0} \frac{1 - \cos x}{x} = 0$$



Viewing the graph.

As we move along the graph of $y = \frac{1 - \cos x}{x}$, shown below, and get closer and closer to $x = 0$, from either the left side or the right side, the functional value seems to get closer and closer to 0.



Hence the graph of $y = \frac{1 - \cos x}{x}$ supports the calculator statement, i.e. that

$$\lim_{x \to 0} \frac{1 - \cos x}{x} = 0$$

(and hence that $\lim_{h \to 0} \frac{1 - \cos h}{h} = 0$).

Note again that whilst the function is undefined for $x = 0$ this does not stop us investigating the behaviour of the function as x gets closer and closer to zero.

Tables of values.

Copy and complete the following tables.

x approaching 0 from the left.

x	$\dfrac{1 - \cos x}{x}$ (values from calculator)
-0·1	
-0·01	
-0·001	
-0·000 1	
-0·000 01	
-0·000 001	

x approaching 0 from the right.

x	$\dfrac{1 - \cos x}{x}$ (values from calculator)
0·1	
0·01	
0·001	
0·000 1	
0·000 01	
0·000 001	

Do your completed tables support the statement

$$\boxed{\lim_{x \to 0} \frac{1 - \cos x}{x} = 0} \quad ?$$

Proof.

This second useful trigonometric limit is proved below. The proof uses the fact, not proved here, that we can write $\lim\limits_{x \to a} \left(\mathrm{f}(x) \times \mathrm{g}(x)\right)$ as $\lim\limits_{x \to a} \mathrm{f}(x) \times \lim\limits_{x \to a} \mathrm{g}(x)$.

$$\begin{aligned}
\lim_{x \to 0} \frac{1 - \cos x}{x} &= \lim_{x \to 0} \left(\frac{1 - \cos x}{x} \; \frac{1 + \cos x}{1 + \cos x}\right) \\
&= \lim_{x \to 0} \frac{1 - \cos^2 x}{x(1 + \cos x)} \\
&= \lim_{x \to 0} \frac{\sin^2 x}{x(1 + \cos x)} \\
&= \lim_{x \to 0} \left(\frac{\sin x}{x} \times \frac{\sin x}{(1 + \cos x)}\right) \\
&= \lim_{x \to 0} \frac{\sin x}{x} \times \lim_{x \to 0} \frac{\sin x}{(1 + \cos x)} \\
&= \quad 1 \quad \times \frac{\sin 0}{1 + \cos 0} \\
&= \quad 0
\end{aligned}$$

Let us now return to statement ★ made on the first page of this chapter:

★ → If $y = \sin x$ then $\dfrac{dy}{dx} = \cos x \lim\limits_{h\to 0} \dfrac{\sin h}{h} - \sin x \lim\limits_{h\to 0} \dfrac{1 - \cos h}{h}$.

Applying our two trigonometric limits:

$$\lim_{h\to 0} \frac{\sin h}{h} = 1 \qquad \text{and} \qquad \lim_{h\to 0} \frac{1 - \cos h}{h} = 0,$$

statement ★ becomes:

$$\text{If } y = \sin x \text{ then } \frac{dy}{dx} = \cos x\,(1) - \sin x\,(0)$$
$$= \cos x$$

Further, if $y = \cos x$ then

$$\frac{dy}{dx} = \lim_{h\to 0} \frac{\cos(x+h) - \cos x}{h}$$
$$= \lim_{h\to 0} \frac{\cos x \cos h - \sin x \sin h - \cos x}{h}$$
$$= \lim_{h\to 0} \frac{\cos x \cos h - \cos x}{h} - \lim_{h\to 0} \frac{\sin x \sin h}{h}$$
$$= -\cos x \lim_{h\to 0} \frac{1 - \cos h}{h} - \sin x \lim_{h\to 0} \frac{\sin h}{h}$$
$$= -\cos x\,(0) - \sin x\,(1)$$
$$= -\sin x$$

Thus

if	y	=	$\sin x$
then	$\dfrac{dy}{dx}$	=	$\cos x$

and

if	y	=	$\cos x$
then	$\dfrac{dy}{dx}$	=	$-\sin x$

Remember • The limit facts used to obtain the above results are true for angles in radians. Thus the above facts again assume radian measure.

The two boxed results above, together with an ability to use

the sum and difference rules, If $y = u \pm v$, $\dfrac{dy}{dx} = \dfrac{du}{dx} \pm \dfrac{dv}{dx}$

the product rule, If $y = u \times v$, $\dfrac{dy}{dx} = v\dfrac{du}{dx} + u\dfrac{dv}{dx}$

➡

the quotient rule, If $y = \frac{u}{v}$, $\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

and the chain rule, If $y = f(u)$ and $u = g(x)$ $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$

which gives us, If $y = [f(x)]^n$ $\frac{dy}{dx} = n\,[f(x)]^{n-1}\,f'(x)$

allow us to differentiate many functions that involve the trigonometric ratios, as the following examples demonstrate.

In each of examples 1 to 6, part (a) reminds you of the rule applied to an expression that does not involve trigonometric functions, and then part (b) applies the rule to a trigonometric expression.

Example 1 (Sum and difference rules.)
Differentiate (a) $x^2 - x^3$ (b) $x^2 + \sin x$

(a) If
$$y = x^2 - x^3$$
$$\frac{dy}{dx} = 2x - 3x^2$$

(b) If
$$y = x^2 + \sin x$$
$$\frac{dy}{dx} = 2x + \cos x$$

Example 2 (Product rule.)
Differentiate (a) $3x^2$ (b) $5 \sin x$

(a) If
$$y = 3x^2$$
$$\frac{dy}{dx} = x^2 \times 0 + 3 \times 2x$$
$$= 6x$$

(b) If
$$y = 5 \sin x$$
$$\frac{dy}{dx} = \sin x \times 0 + 5 \times \cos x$$
$$= 5 \cos x$$

Example 3 (Product rule.)
Differentiate (a) $(3x - 4)(5x^2 + 3)$ (b) $(2 - \cos x)(1 + \sin x)$

(a) If
$$y = (3x - 4)(5x^2 + 3)$$
$$\frac{dy}{dx} = (5x^2 + 3) \times 3 + (3x - 4) \times 10x$$
$$= 15x^2 + 9 + 30x^2 - 40x$$
$$= 45x^2 - 40x + 9$$

(b) If $y = (2 - \cos x)(1 + \sin x)$

$$\frac{dy}{dx} = (1 + \sin x) \times (\sin x) + (2 - \cos x) \times \cos x$$

$$= \sin x + \sin^2 x + 2\cos x - \cos^2 x$$

Example 4 (Quotient rule.)

Differentiate (a) $\dfrac{x^2 - 1}{2x - 3}$ (b) $\dfrac{\sin x}{\cos x}$

(a) If $y = \dfrac{x^2 - 1}{2x - 3}$

$$\frac{dy}{dx} = \frac{(2x-3)(2x) - (x^2-1)(2)}{(2x-3)^2}$$

$$= \frac{2x^2 - 6x + 2}{(2x-3)^2}$$

(b) If $y = \dfrac{\sin x}{\cos x}$

$$\frac{dy}{dx} = \frac{(\cos x)(\cos x) - (\sin x)(-\sin x)}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$

$$= \frac{1}{\cos^2 x}$$

Note
- From the result above it follows that if $y = \tan x$, $\dfrac{dy}{dx} = \dfrac{1}{\cos^2 x}$.
- Writing $\dfrac{1}{\cos x}$ as $\sec x$ we could write: If $y = \tan x$, $\dfrac{dy}{dx} = \sec^2 x$.

  However, whilst the term "sec x" (and cosec x and cot x) will be familiar to students who followed *Unit Two* of *Mathematics Specialist* it is not required knowledge for this unit.

Example 5 (Chain rule.)

(a) By letting $u = 2x - 7$ determine the derivative of $3(2x-7)^5$ using the chain rule.
(b) By letting $u = 2x + 1$ determine the derivative of $\sin(2x+1)$ using the chain rule.

(a) If $y = 3(2x-7)^5$ then with $u = (2x-7)$, $y = 3u^5$.

Hence $\frac{du}{dx} = 2$ and $\frac{dy}{du} = 15u^4$

Thus, by the chain rule
$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du}\frac{du}{dx}\\ &= 15u^4 \times 2\\ &= 30u^4\\ &= 30(2x-7)^4\end{aligned}$$

(b) If $y = \sin(2x+1)$ then with $u = (2x+1)$, $y = \sin u$.

Hence $\frac{du}{dx} = 2$ and $\frac{dy}{du} = \cos u$

Thus, by the chain rule
$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du}\frac{du}{dx}\\ &= (\cos u) \times 2\\ &= 2\cos u\\ &= 2\cos(2x+1)\end{aligned}$$

As is probably already the case with non trigonometrical expressions, the reader will, with practice, be able to differentiate expressions like $3(2x - 7)^5$ and $\sin(2x + 1)$ directly, without formally using the chain rule, as in the next examples.

Example 6 (Chain rule.)

Differentiate (a) $(2x+3)^4$ (b) $\sin^4 x$

(a) If $y = (2x+3)^4$
$$\begin{aligned}\frac{dy}{dx} &= 4(2x+3)^3\,(2)\\ &= 8(2x+3)^3\end{aligned}$$

(b) If $y = \sin^4 x$
$$\begin{aligned}\frac{dy}{dx} &= 4(\sin x)^3 \cos x\\ &= 4\sin^3 x \cos x\end{aligned}$$

Example 7 (Chain rule.)

Differentiate (a) $\cos(2x+3)$ (b) $\sin(5-2x)$

(a) If $y = \cos(2x+3)$

$$\frac{dy}{dx} = -2\sin(2x+3)$$

Thoughts:
"cos" differentiates to "$-\sin$".
$2x+3$ differentiates to 2.
Answer: $(2)(-\sin(2x+3))$

(b) If $y = \sin(5-2x)$

$$\frac{dy}{dx} = -2\cos(5-2x)$$

Thoughts:
"sin" differentiates to "cos".
$5-2x$ differentiates to -2.
Answer: $(-2)(\cos(5-2x))$

Note • You are already familiar with using your calculator to obtain derivatives. Whilst you are encouraged to use this facility when appropriate, make sure you can differentiate trigonometrical functions without the assistance of a calculator when required to do so. Also be aware that because of the various trigonometrical identities that exist, and because of the method of display, the calculator answer may sometimes, at first glance, appear different to the expression you obtain.

For example, asked to determine

$$\frac{d}{dx}(\sin x\cos x),$$

for which we might use the product rule and write the answer as

$$\cos x\cos x + \sin x(-\sin x)$$

i.e. $$\cos^2 x - \sin^2 x$$

a calculator may give an equally correct, but rather different looking answer, as shown below.

$$\frac{d}{dx}\left(\sin(x)\cdot\cos(x)\right)$$

$$2\cdot\left(\cos(x)\right)^2 - 1$$

Exercise 7A

Find the derivatives, with respect to x, of each of the following

Sum and difference rules.

1. $x^5 - x^2$
2. $3 + x^3$
3. $5 - \cos x$
4. $\sin x - \cos x$
5. $\cos x - \sin x$
6. $x - \tan x$

Product rule.

7. $(x + 1)(2x - 3)$
8. $5x^2(1 - 5x)$
9. $6 \sin x$
10. $4 \cos x$
11. $x \sin x$
12. $x^2 \cos x$

Quotient rule.

13. $\dfrac{x}{3x^2 - 1}$
14. $\dfrac{x^2 + 1}{x^2 - 1}$
15. $\dfrac{\cos x}{x}$
16. $\dfrac{\sin x}{x}$
17. $\dfrac{x}{\sin x}$
18. $\dfrac{x}{\cos x}$

Chain rule.

Find $\dfrac{dy}{dx}$, in terms of x, for each of the following.

19. $y = 3u^2 - 5$ and $u = x^2 + 1$.
20. $y = \sqrt{u}$ and $u = x^2 - 1$.
21. $y = \sin u$ and $u = 6x$.
22. $y = \cos u$ and $u = 2x + 3$.
23. $y = \sin^2 x$
24. $y = \sin^3 x$
25. $y = \cos^5 x$
26. $y = \cos 3x$
27. $y = \sin(3x - 7)$
28. $y = \cos(2x + 5)$

Miscellaneous.

Differentiate each of the following with respect to x.

29. $2 - 3\cos x$
30. $3x + 2\cos x$
31. $\sin 2x$
32. $x^2 - \cos x$

33. $\dfrac{1 + \sin x}{x^2}$

34. $3\sin x - 2\cos x$

35. $\cos 3x$

36. $\cos 9x$

37. $3\cos 2x$

38. $5\sin 3x$

39. $2\sin 3x + 3\cos 2x$

40. $\sin^5 x$

41. $5\cos^2 x$

42. $\sqrt{\sin x}$

Determine $f'(x)$ for each of the following.

43. $f(x) = \sin 7x$

44. $f(x) = \sin 8x$

45. $f(x) = \sin 4x + \cos 4x$

46. $f(x) = 2\sin(3x - 1)$

47. $f(x) = 4\cos(4x + 3)$

48. $f(x) = 2\sin^3 x$

49. $f(x) = 3\cos^2 x$

50. $f(x) = x\cos x$

51. $f(x) = x^2\cos x$

52. $f(x) = 2x\sin x$

53. $f(x) = \dfrac{2\sin x}{\cos x}$

54. $f(x) = 2\tan x$

Determine the gradient of each of the following at the given point.

55. $y = \sin x$ at the point $\left(\dfrac{\pi}{6}, \dfrac{1}{2}\right)$.

56. $y = \cos 2x$ at the point $\left(\dfrac{\pi}{6}, \dfrac{1}{2}\right)$.

57. $y = 2\sin x\cos x$ at the point $(0, 0)$.

58. $y = 3\sin^2 x$ at the point $(\pi, 0)$.

Determine $\dfrac{d^2y}{dx^2}$ for each of the following.

59. $y = \sin x$

60. $y = \cos 5x$

61. $y = 3\sin 2x$

62. $y = \sin x + \cos x$

63. Find the equation of the tangent to the curve $y = x\sin x$ at the point $(\pi/2, \pi/2)$.

64. Find the equation of the tangent to $y = x + 3\cos 2x$ at the point $(0, 3)$.

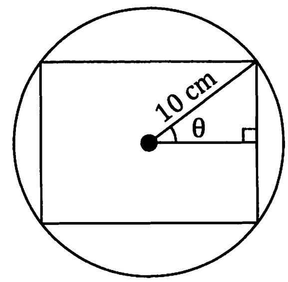
65. If $f(x) = \sin 2x$ find exact values for (a) $f'(\pi/6)$ (b) $f''(\pi/6)$.

66. If $y = \sin x^\circ$ find $\frac{dy}{dx}$. (Hint: $1^\circ = \frac{\pi}{180}$ radians.)

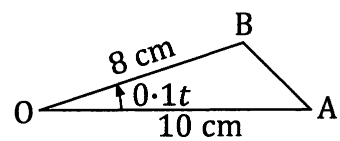
67. With θ as shown in the diagram, and $0 < \theta < \frac{\pi}{2}$, show that the area of a rectangle drawn with all four vertices touching a circle of radius 10 cm, is A cm^2 where

$$A = 400 \sin\theta \cos\theta.$$

<u>Use calculus</u> to prove that the rectangle drawn in this way, and having maximum area, will be a square, and determine its area and side length.



68. Triangle OAB has OA = 10 cm, OB = 8 cm and $\angle BOA = 0{\cdot}1t$ radians, where t is the time in seconds. Thus when $t = 0$, OB lies along OA and as t increases $\angle AOB$ "opens".
Find an expression in terms of t for the rate of change of the area of ΔOAB with respect to time.



Determine the instantaneous rate of change in the area of ΔOAB with respect to time when (a) $t = 1$,
(b) $t = 5$,
(c) $t = 10$,
(d) $t = 20$.
(Give answers in cm^2/sec and correct to two decimal places.)

69. Given that $x = 5\sin(3t)$, $t \geq 0$:

(a) Find the maximum value of x and the smallest value of t for which it occurs.

(b) Find the three smallest values of t for which $x = 2{\cdot}5$.

(c) Find, correct to 1 decimal place, $\frac{dx}{dt}$ when $t = 0{\cdot}6$.

(d) Prove that $\frac{d^2x}{dt^2} = kx$, and find k.

70. Use calculus to determine, correct to 4 decimal places, the value of θ, for $0 \leq \theta \leq \pi/2$, which maximises

$$3\sin\theta + 4\cos\theta,$$

and find this maximum value.
(Note: Students who followed *Unit Two* of *Mathematics Specialist*, may remember solving this sort of optimisation question there without using calculus.)

Antidifferentiation of functions involving sine and cosine.

Our ability to differentiate expressions involving trigonometric functions means that our antidifferentiation, or integration, can now also involve functions of this type. The following examples, and the exercise that follows, involve determining antiderivatives of trigonometrical functions.

Example 8

Antidifferentiate (a) $\cos x$, (b) $\sin x$.

(a) We know that if $y = \sin x$

then $\dfrac{dy}{dx} = \cos x.$

Thus $\displaystyle\int \cos x \, dx = \sin x + c$

The antiderivative is $\sin x + c$.

(b) We know that if $y = \cos x$

then $\dfrac{dy}{dx} = -\sin x.$

Thus $\displaystyle\int \sin x \, dx = -\cos x + c$

The antiderivative is $-\cos x + c$.

$$\int \cos(x)\, dx$$

$$\sin(x)$$

$$\int \sin(x)\, dx$$

$$-\cos(x)$$

Note • If $y = \sin f(x)$ then, by the chain rule, $\dfrac{dy}{dx} = f'(x) \cos f(x)$.

Hence

$$\boxed{\int f'(x) \cos f(x)\, dx = \sin f(x) + c}$$

Similarly

$$\boxed{\int f'(x) \sin f(x)\, dx = -\cos f(x) + c}$$

- In the following examples, some show the method of making an intelligent first guess and then making suitable adjustments and some show that of making an initial rearrangement to aid the antidifferentiation process. With practice some of the antiderivatives can be written directly by "mentally juggling" an intelligent first guess.
- The antiderivative of $\tan x$ is beyond the requirements of this unit.
- In some cases rearranging the given expression using one of the trigonometric identities can be useful, see example 12.

Example 9

Find the antiderivative of (a) $8 \sin 2x$, (b) $8 \cos (2x + 1)$, (c) $\cos 5x + \sin 2x$.

(a) Try $y = \cos 2x$

then, by the chain rule, $\frac{dy}{dx} = (2)(-\sin 2x)$

$= -2 \sin 2x$

Hence if $y = -4 \cos 2x$

then $\frac{dy}{dx} = 8 \sin 2x$, as required.

The required antiderivative is $-4 \cos 2x + c$.

(b) Try $y = \sin (2x + 1)$

then, by the chain rule $\frac{dy}{dx} = 2 \cos (2x + 1)$

Hence the required antiderivative is $4 \sin (2x + 1) + c$.

(c) Try $y = \sin 5x - \cos 2x$

then $\frac{dy}{dx} = 5 \cos 5x + 2 \sin 2x$

Hence the required antiderivative is $\frac{1}{5} \sin 5x - \frac{1}{2} \cos 2x + c$.

Example 10

Find the antiderivative of (a) $3 \sin x$, (b) $15 \cos 5x$.

(a)
$$\begin{aligned} \int 3 \sin x \, dx &= 3 \times \int \sin x \, dx \\ &= 3 \times (-\cos x) + c \\ &= -3 \cos x + c. \end{aligned}$$

The required antiderivative is $-3 \cos x + c$.

(b) (First note that the derivative of $5x$ is 5.)
$$\begin{aligned} \int 15 \cos 5x \, dx &= 3 \times \int 5 \times \cos 5x \, dx \\ &= 3 \times \sin 5x + c \\ &= 3 \sin 5x + c \end{aligned}$$

The required antiderivative is $3 \sin 5x + c$.

Example 11

Find the antiderivative of (a) $\cos^4 x \sin x$, (b) $3 \sin^3 4x \cos 4x$.

(a) Try $y = \cos^5 x$

then $\frac{dy}{dx} = 5 \cos^4 x (-\sin x)$

$= -5 \cos^4 x \sin x$

Thus if $y = -\frac{1}{5} \cos^5 x$

$\frac{dy}{dx} = \cos^4 x \sin x$ as required.

The required antiderivative is $-\frac{1}{5} \cos^5 x + c$.

(b) (Note that the derivative of $\sin 4x$ is $4 \cos 4x$. By the rearrangement approach we attempt to set up an expression of the form $f'(x)\,[f(x)]^n$).

$$\int 3 \sin^3 4x \cos 4x \, dx = \frac{3}{4} \times \int (4 \cos 4x) \times \sin^3 4x \, dx$$

$$= \frac{3}{4} \times \frac{\sin^4 4x}{4} + c$$

$$= \frac{3 \sin^4 4x}{16} + c$$

The required antiderivative is $\frac{3}{16} \sin^4 4x + c$.

Example 12

Find the antiderivative of $\sin 4x \cos 3x + \cos 4x \sin 3x$.

Remembering that $\sin(A + B) = \sin A \cos B + \cos A \sin B$ it follows that

$$\sin 4x \cos 3x + \cos 4x \sin 3x = \sin(4x + 3x)$$

$$= \sin 7x$$

Hence $\int (\sin 4x \cos 3x + \cos 4x \sin 3x)\, dx = \int \sin 7x \, dx$

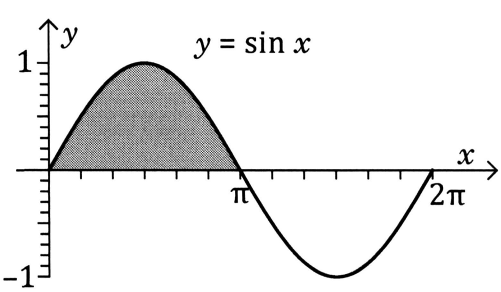
$$= -\frac{1}{7} \cos 7x + c$$

The required antiderivative is $-\frac{1}{7} \cos 7x + c$.

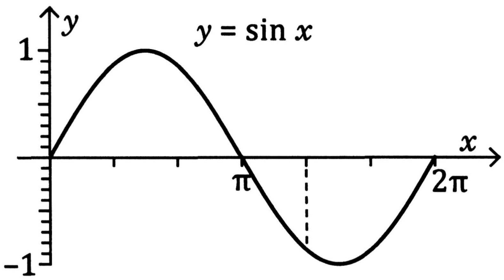
Exercise 7B

Find the antiderivative of each of the following.

1. $5 \cos x$
2. $2 \sin x$
3. $-10 \sin x$
4. $-2 \cos x$
5. $6 \cos 2x$
6. $2 \cos 6x$
7. $12 \sin 4x$
8. $-\sin 3x$
9. $-8 \cos 10x$
10. $\sin \frac{x}{2}$
11. $\cos \frac{3x}{2}$
12. $-6 \sin \frac{2x}{3}$
13. $6 \sin (2x + 3)$
14. $3 \cos (2x - 3)$
15. $\cos\left(2x + \frac{2\pi}{3}\right)$
16. $\sin (-x)$
17. $\frac{4}{\cos^2 x}$ (Hint: What is the derivative of $\tan x$?)
18. $6 \cos 2x + 6 \sin 3x$
19. $\cos 8x - 4 \sin 2x$
20. $2x + 4 \cos x + 6 \cos 2x$
21. $3 + 4x - 6x^2 + 10 \cos 5x - 2 \sin 4x$
22. $\cos^3 x \sin x$
23. $30 \cos^5 x \sin x$
24. $\sin 5x \cos 2x + \cos 5x \sin 2x$
25. $\sin 3x \cos x - \cos 3x \sin x$
26. $\cos 5x \cos 2x - \sin 5x \sin 2x$
27. $\cos 5x \cos x + \sin 5x \sin x$
28. Evaluate $\int_0^{\pi/2} \sin x \, dx$
29. Evaluate $\int_0^{\pi/2} \cos x \, dx$
30. Evaluate $\int_{\pi/2}^{\pi} \cos \frac{x}{2} \, dx$
31. Evaluate (a) $\int_0^{\pi/4} \sin x \, dx$, (b) $\int_{\pi/4}^{0} \sin x \, dx$.
32. Find the area enclosed by $y = \sin x$ and the x-axis from $x = 0$ to $x = \pi$.



33. Find the area between $y = \sin x$ and the x-axis from:
    (a) $x = \pi$ to $x = 4\pi/3$,
    (b) $x = 0$ to $x = 4\pi/3$.



34. A particle moves along a straight line such that its velocity, at time t seconds, is given by v metres per second, where

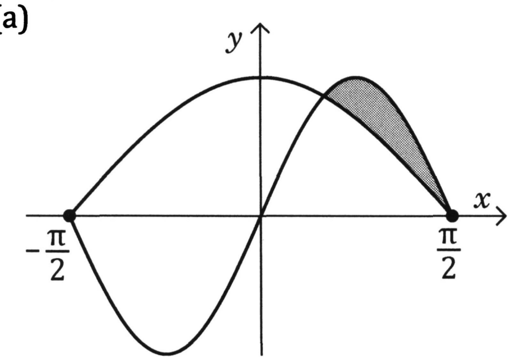
$$v = 2 \cos 2t.$$

When $t = 0$ the displacement of the particle from an origin O is 5 metres.
Find (a) the greatest speed of the particle,
(b) an expression for the displacement of the particle from O at time t,
(c) the least distance the particle is from the origin,
(d) an expressions for the acceleration of the particle at time t.

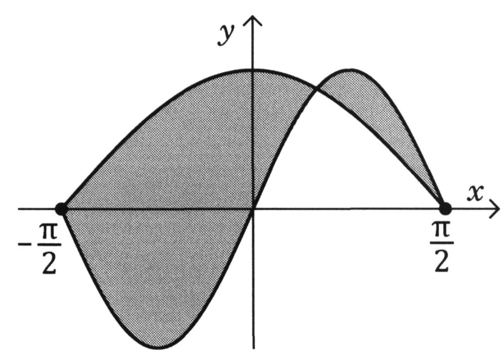
35. Each of the graphs below show $y = \sin 2x$
and $y = \cos x$, each for $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$.

Use your calculator to determine the area shaded in each case.

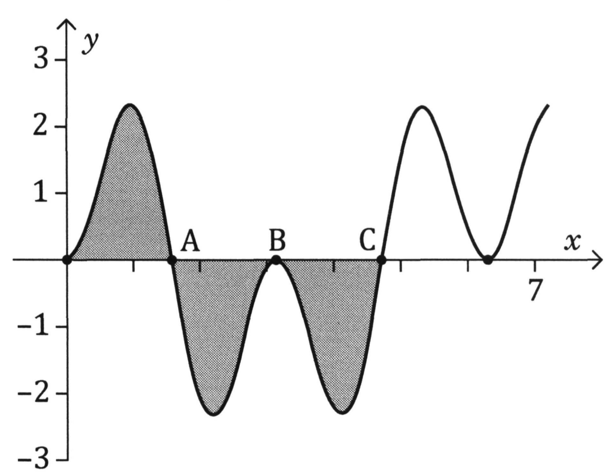
(a)



(b)



36. The graph of $y = 6 \cos x \sin^2 x$ is shown on the right for $0 \le x \le 7$.
    (a) Find the exact coordinates of points A, B and C, all of which lie on the x-axis.
    (b) With the assistance of your calculator if you wish, find the total shaded area.



Miscellaneous Exercise Seven.

This miscellaneous exercise may include questions involving the work of this chapter, the work of any previous chapters, and the ideas mentioned in the preliminary work section at the beginning of the book.

Differentiate the following with respect to x.

1.	e^x	2.	$2e^x$	3.	$8e^x$
4.	$e^x + \sin x$	5.	$e^{\cos x}$	6.	$e^{\sin 2x}$
7.	$e^{2\sin x}$	8.	$e^x + \frac{1}{x}$	9.	$4(\sqrt{x}) + e^{3x}$
10.	$e^x\sqrt{x}$	11.	$e^x \sin x$	12.	$e^x \cos 2x$
13.	$e^x \sin^2 x$	14.	e^{3x^2+2}	15.	$e^{x^2+\sin x}$

16. If $T = (2r + 3)^3$ find, without the assistance of a calculator, an expression for the rate of change of T with respect to r.

17. Evaluate the following definite integrals "by hand" and then use your calculator to check your answers.

    (a) $\int_0^2 4e^{2x}\,dx$ (b) $\int_2^5 \frac{1}{x^2}\,dx$ (c) $\int_1^2 30(2x-3)^4\,dx$

18. Find the exact values of x, for $-2\pi \le x \le 2\pi$, for which the gradient of the curve

$$y = e^x \sin x$$

is zero.

19. If $y = e^{-x}\sin x$ find $\frac{dy}{dx}$ when $x = \pi$.

20. Write down an expression of the form ax^n for $\lim_{h\to 0}\left(\frac{\sqrt{x+h}-\sqrt{x}}{h}\right)$.

    (This should not require much, if any, working.)

21. It is expected that 3 000 000 tonnes of a particular resource is to be available for use this year but the amount available in t years' time is expected to be A tonnes where

$$A = 3\,000\,000\, e^{-0\cdot 1t}.$$

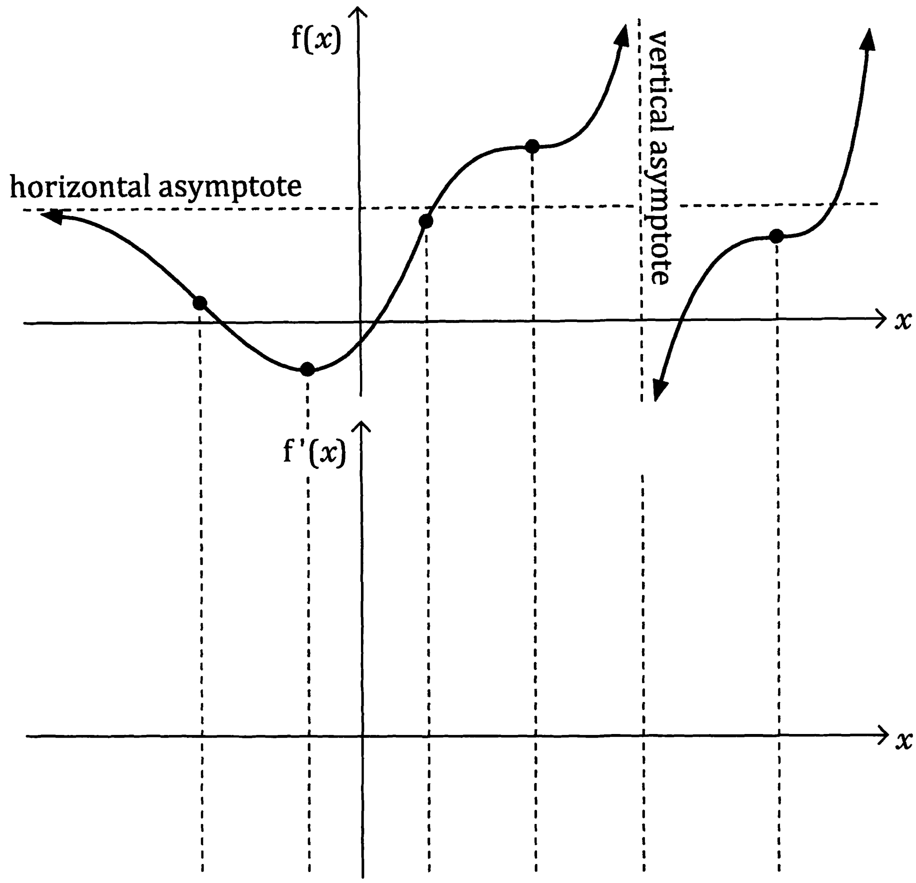
Is A increasing or decreasing as t increases?

Find the value of $\frac{dA}{dt}$, in tonnes/year, when (a) $t = 2$,

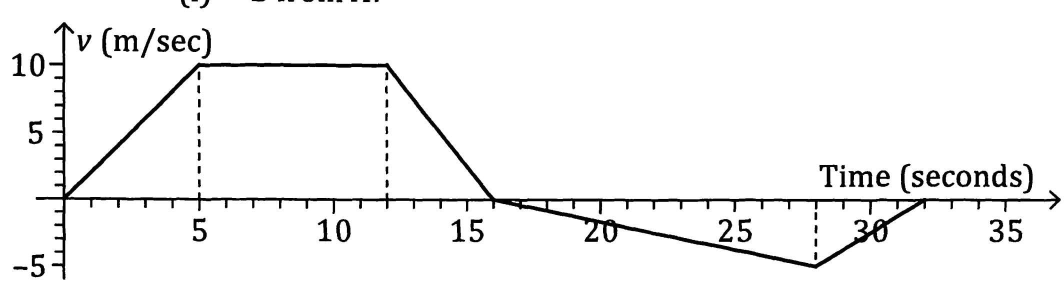
(b) $t = 5$,

(c) $t = 10$.

22. Copy the following graph of $y = f(x)$ and complete the graph of $f'(x)$. (Pay particular regard to the points marked •).
Remember: An **asymptote** is a line that a curve gets closer and closer to without ever quite touching.



23. The velocity-time graph shown below is for a particle moving in a straight line, from rest at A, through B to C and then back to rest at B.
(a) What is the velocity of the particle 8 seconds after leaving A?
(b) What is the acceleration of the particle 13 seconds after leaving A?
(c) How far does the particle move in the first 28 seconds?
(d) What is the particle's displacement from A 28 seconds after leaving A?
How far is (e) C from A,
(f) B from A?



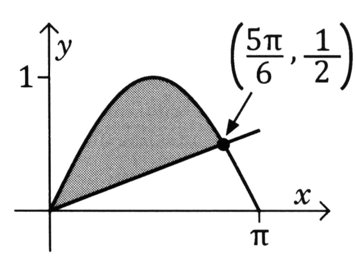
24. One suggestion for solving the water shortage in certain parts of the world involved towing icebergs from the polar regions to other parts of the world. Clearly some ice would melt on the journey but enough might be left to make the journey worthwhile, especially if the iceberg was "lagged" in some way. To investigate this situation mathematically we could consider the iceberg to be roughly spherical with an initial radius of 100 m and assume that the initial radius reduces by 3 m each day of travel.
    (a) Find a formula for the volume, V m^3 ($V \geq 0$), of the iceberg after x days of travel.
    (b) On which day will the volume have reduced to half the original amount?
    (c) Find a formula that will allow the rate at which the volume is changing to be determined (in m^3/day) given x.
    (d) What is the rate of volume loss, in m^3/day, when $x = 5$?
    (e) The idea that the radius would reduce by 3 metres per day was not felt to be realistic because, as the journey progressed, the iceberg would be towed into increasingly warmer surroundings. An alternative formula could be:

    $$V = \frac{4}{3}\pi\,(100 - 2x - x^2)^3 \qquad \text{for } V \geq 0.$$

    Find a formula for the instantaneous rate of volume loss now and again evaluate it for $x = 5$.

25. An object placed in a particular fluid sinks such that, t seconds after release, its downward velocity is v m/sec where $v = 2(1 - e^{-0\cdot 2t})$ m/sec.
    Find, correct to two decimal places,
    (a) the speed of the object 2 seconds after release,
    (b) the downward acceleration of the object 2 seconds after release,
    (c) the downward acceleration of the object 10 seconds after release.

26. The diagram shows the line $y = \dfrac{3x}{5\pi}$ and the curve $y = \sin x$ for $0 \leq x \leq \pi$.



    (a) Find as an exact value the enclosed area shown shaded in the diagram.
    (b) View the line and the curve on your calculator and hence write an exact answer for the total area <u>enclosed</u> between the line and the curve for unrestricted values of x.